Math 2A – Vector Calculus – Fall ’07 – Chapter 10 Test   Name____________________

Show your work for credit. Do not use a calculator. Write all responses on separate paper.

1. Find a parameterization of the line segment from $\vec{r}(2)$ to $\vec{r}(5)$ where $\vec{r}(t) = \left\langle \cos(\pi t), t^2, \sin\left(\frac{\pi}{2}t\right)\right\rangle$.

2. Find a vector function that represents the curve of intersection of the elliptical cylinder $x^2 + 4y^2 = 1$ with the plane $x + y + z = 1$.

3. Find the length of the curve $\vec{r}(t) = \langle 3\sin t, 3t^{3/2}, -3\cos t\rangle$ on the interval $0 \le t \le \pi$.

4. Reparameterize the curve $\vec{r}(t) = \langle e^{2t}, e^{2t}\sin 2t, e^{2t}\cos 2t\rangle$, with respect to arc length measured from the point (1,0,1) in the direction of increasing t.

5. Find an equation for the osculating plane of the curve $\vec{r}(t) = \langle 2t, 3t, t^3 + 3t\rangle$ at the point (2, 3, 4).

6. A particle starts at (0,0,100) and has initial velocity $\langle 1, 2, 3\rangle$ and moves with acceleration $\vec{a}(t) = \langle -0.01, 0, -8\rangle$. Find its position function and determine where it intersects the xy plane.

7. Find an equation for the osculating circle at the vertex of the parabola $y = 2 - \frac{1}{2}(x-1)^2$.

8. Find the tangential and normal components for the acceleration of a particle whose position function is $\vec{r}(t) = \left\langle \int_0^t \sin(\pi u^2)\,du, \int_0^t \cos(\pi u^2)\,du, 1\right\rangle$.

9. Describe the surface: $\vec{r}(u,v) = \left\langle 1 + \sin u, 2 + \cos u, \frac{1}{2}(1 + \sin v)\right\rangle$

10. Find a parametric representation for the part of the hyperboloid $(x-1)^2 + y^2 - z^2 = 1$ that lies to the right of the plane $x = 1$.

Math 2A – Vector Calculus – Fall ’07 – Chapter 10 Test Solutions.

1. Find a parameterization of the line segment from $\vec r(2)$ to $\vec r(5)$ where $\vec r(t) = \left\langle \cos(\pi t), t^2, \sin\left(\frac{\pi}{2} t\right)\right\rangle$.

   ANS: $\vec p(t) = \vec r(2) + \left(\vec r(5) - \vec r(2)\right)t = \langle 1,4,0\rangle + \left(\langle -1,25,1\rangle - \langle 1,4,0\rangle\right)t = \langle 1-2t, 4+21t, t\rangle$

2. Find a vector function that represents the curve of intersection of the elliptical cylinder $x^2 + 4y^2 = 1$ with the plane $x + y + z = 1$.

   ANS: $\vec r(t) = \left\langle \cos(3t), \frac{1}{2}\sin(3t), 1 - \cos(3t) - \frac{1}{2}\sin(3t)\right\rangle$ will do the trick.

3. Find the length of the curve $\vec r(t) = \langle 3\sin t, 3t^{3/2}, -3\cos t\rangle$ on the interval $0 \le t \le \pi$.

   ANS: $\int_0^\pi \sqrt{9\cos^2 t + \frac{81}{4}t + 9\sin^2 t}\,dt = \frac{3}{2}\int_0^\pi \sqrt{4+9t}\,dt = \left.\frac{(4+9t)^{3/2}}{9}\right|_0^\pi = \frac{(4+9\pi)^{3/2} - 8}{9}$

4. Reparameterize the curve $\vec r(t) = \langle e^{2t}, e^{2t}\sin 2t, e^{2t}\cos 2t\rangle$, with respect to arc length measured from the point (1,0,1) in the direction of increasing t.

   ANS: $s(t) = \int_0^t \sqrt{4e^{4u} + \left(2e^{2u}\sin 2t + 2e^{2u}\cos 2t\right)^2 + \left(2e^{2u}\cos 2t - 2e^{2u}\sin 2t\right)^2}\,du =$

   $= \int_0^t \sqrt{4e^{4u} + 4e^{4u} + 4e^{4u}}\,du = 2\sqrt{3}\int_0^t e^{2u}\,du = \sqrt{3}\left(e^{2t} - 1\right)$

   Thus $2t = \ln\left(\frac{s}{\sqrt{3}} + 1\right)$ and substituting for $2t$ we have

   $\vec r(s) = \left\langle \frac{s}{\sqrt{3}} + 1, \left(\frac{s}{\sqrt{3}} + 1\right)\sin\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right), \left(\frac{s}{\sqrt{3}} + 1\right)\cos\left(\ln\left(\frac{s}{\sqrt{3}} + 1\right)\right)\right\rangle$

5. Find an equation for the osculating plane of the curve $\vec r(t) = \langle 2t, 3t, t^3 + 3t\rangle$ at the point (2, 3, 4).

   ANS: $\vec r\,'(1) = \langle 2,3,6\rangle$ and $\vec r\,''(1) = \langle 0,0,6\rangle$ are both in the osculating plane, so their cross product $\vec r\,'(1) \times \vec r\,''(1) = \langle 0,0,6\rangle \times \langle 2,3,6\rangle = \langle -18,12,0\rangle = -6\langle 3,-2,0\rangle$ is normal and choosing d in the equation $3x - 2y = d$ to fit the given point we have $3x - 2y = 0$

6. A particle starts at (0,0,100) and has initial velocity $\langle 1,2,3\rangle$ and moves with acceleration $\vec a(t) = \langle -0.01, 0, -8\rangle$. Find its position function and determine where it intersects the *xy* plane.

   ANS: The velocity is $\vec v(t) = \int_0^t \vec a(u)\,du + \vec v(0) = \langle 1 - 0.01t, 2, 3 - 8t\rangle$ and so the position is

$\vec{r}(t) = \int_0^t \vec{v}(u)\,du + \vec{r}(0) = \left\langle t - 0.005t^2, 2t, 100 + 3t - 4t^2 \right\rangle$ Thus it'll hit the ground when

$100 + 3t - 4t^2 = 0 \Leftrightarrow t^2 - \frac{3}{4}t = 25 \Leftrightarrow \left(t - \frac{3}{8}\right)^2 = 25 + \frac{9}{64} \Leftarrow t = \frac{3 + \sqrt{1609}}{8}$

7. Find an equation for the osculating circle at the vertex of the parabola $y = 2 - \frac{1}{2}(x-1)^2$.

   ANS: Taking $\vec{r}(t) = \left\langle t, 2 - \frac{1}{2}(t-1)^2, 0 \right\rangle \Rightarrow \vec{r}\,'(t) = \langle 1, -(t-1), 0 \rangle \Rightarrow \vec{r}\,''(t) = \langle 0, -1, 0 \rangle$ we have at the vertex, $\vec{r}\,'(1) = \langle 1, 0, 0 \rangle$ and

   $\vec{r}\,' \times \vec{r}\,'' = \langle 1, 0, 0 \rangle \times \langle 0, -1, 0 \rangle = \langle 0, 0, -1 \rangle$

   $\kappa = \left|\frac{d\hat{T}}{ds}\right| = \frac{|\hat{T}\,'|}{|r\,'|} = \frac{|r\,'|^2 |\hat{T}||\hat{T}\,'|}{|r\,'|^3} = \frac{|r\,'|^2 |\hat{T} \times \hat{T}\,'|}{|r\,'|^3} = \frac{\left||r\,'|\hat{T} \times |r\,'|\hat{T}\,'\right|}{|r\,'|^3} = \frac{|r\,' \times r\,''|}{|r\,'|^3} = 1$ so the radius is the reciprocal = 1 and the center is (0,1,0) and the equation is $(x-1)^2 + (y-1)^2 = 1$.

8. Find the tangential and normal components for the acceleration of a particle whose position function is $\vec{r}(t) = \left\langle \int_0^t \sin(\pi u^2)\,du, \int_0^t \cos(\pi u^2)\,du, 1 \right\rangle$.

   ANS: $\vec{v}(t) = \vec{r}\,'(t) = \left\langle \sin(\pi t^2), \cos(\pi t^2), 0 \right\rangle = \hat{T}(t)$. Now $s(t) = \int_0^t |r\,'(u)|\,du = \int_0^t du = t$ so $a_N = \kappa v^2 = \left|\frac{d\hat{T}}{ds}\right| |\vec{r}\,'(t)|^2 = \sqrt{(2\pi t \cos(2\pi t))^2 + (-2\pi t \sin(2\pi t))^2} \cdot 1^2 = 2\pi t$ and

   $a_T = \frac{d}{dt}|\vec{r}\,'(t)| = \frac{d}{dt}1 = 0$.

9. Describe the surface: $\vec{r}(u,v) = \left\langle 1 + \sin u, 2 + \cos u, \frac{1}{2}(1 + \sin v) \right\rangle$

   ANS: This is the portion of the cylinder $(x-1)^2 + (y-2)^2 = 1$ of radius 1 with axis $\vec{r}(t) = \langle 1, 2, t \rangle$ between $0 \le z \le 1$.

10. Find a parametric representation for the part of the hyperboloid $(x-1)^2 + y^2 - z^2 = 1$ that lies to the right of the plane $x = 1$.

    ANS: $x = 1 + \sqrt{1 - y^2 + z^2}$ can be parameterized simply by

    $\vec{r}(u,v) = \left\langle 1 - \sqrt{1 - u^2 + v^2}, u, v \right\rangle$